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DYNAMIC STABILITY OF A THREE-DIMENSIONAL STRING SUBJECTED TO BOTH MAGNETIC AND TENSIONED EXCITATIONS

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1. INTRODUCTION

In a previous paper [1], the dynamic stability of a moving string in three-dimensional vibration was studied. It was shown that the unstable regions would be shifted to the lower frequency if the string transport speed increased and to the higher frequency if the wave propagation speed increased. The non-linear vibrations of an axially moving string were extensively studied [2–8]. The effect of magnetic field was not considered in these studies.

The interaction of electromagnetic fields with deformable media is closely related to some fields of modern technology and nuclear physics. Moon and Pao [9, 10] showed that the natural frequency of a beam–plate in a transverse magnetic field caused a buckled phenomenon when it reached a critical value. Lu *et al*. [11] analyzed a model of a magnetoelastic buckled beam subjected to an external axial periodic force in a periodic transverse magnetic field. Wolfe and Seidman [12–14] studied a series of problems of the hyperelastic conducting rod undergoing flexure, tension, shear, and extension in a parallel magnetic field. In the symmetric case, which admits trivial solution, they proved that in the trivial state, non-trivial solutions can exist if the field is strong enough. In the physical sense, the trivial configuration of the system will cause the rod to remain straight and untwisted. However, the rod is bent and twisted if the field is strong enough. To the author's knowledge, there is no paper concerning the dynamic stability of a moving string in an alternating magnetic field.

This paper presents the dynamic stability of an axially moving string subjected to both actions of an alternating uniform transverse magnetic field and a periodic tension force. Owing to time-varying actions, terms with time-dependent coefficients appear in the equations of motion which results in the existence of parametric instability. First, Galerkin's procedure is applied to discretize the continuous system into a finite-degree-of-freedom system. Then, the equations of the discrete system are decoupled by using a special modal analysis procedure [1, 15] which is suitable for the gyroscopic systems. Finally, the first-order simultaneous differential equations are solved by the method of multiple-scales. The system response and the expressions for the boundaries of the unstable regions are obtained. It is well known that the unstable regions occurring at the higher frequencies are better than those at the lower frequencies for most mechanical applications. It is found in the present paper that the unstable regions will be shifted to the lower frequencies when the string transport speed increases or the wave propagation speed decreases.

2. EQUATION OF MOTION

A moving string passing through two fixed eyelets is modeled as shown in Figure 1. The co-ordinate system ($oxyz$) is used to interpret the overall phenomenon of the system. The values $u(x, t)$, $v(x, t)$, and $w(x, t)$ are the displacement components of an arbitrary

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point P in the x, y and z directions, respectively. One assumes the moving string subjected to both external perturbations, an alternating uniform transverse magnetic field $B = (B_0 + B_1 \cos \Omega_{mf} t)j$ in the y direction, and a periodic tension force $T = (T_0 + T_1 \cos \Omega_{tf} t)i$ in the x direction. The definitions of the symbols are the same as those in [1, 11].

Under the *quasi-static assumption* [10], the perturbed magnetic field caused by the deformation at any instant is the same as that in the static case for the instantaneous configuration. The frequency of the applied time-dependent magnetic field is assumed to be low enough and the string diameter is sufficiently small, and hence the skin effect on the magnetic field can be ignored.

The most general form of Hamilton's principle is

$$\int_{t_1}^{t_2} (\delta L + \delta W)\, \mathrm{d}t = 0, \tag{1}$$

where δL is the variation of Lagrangian density which is the same as that reported in Huang *et al*. [1]. δW is the total virtual work, and $\delta W = \delta W_f + \delta W_m$. δW_f and δW_m are the virtual work respectively done by the viscous damping force and the external magnetic field on the system.

By using the following dimensionless variables and parameters $U = u/l$, $V = v/l$, $W = w/l$, $\xi = x/l$, $\tau = c_2 t/l$, $\bar{T} = T_1/T_0$, $\bar{B} = B_1/B_0$, $\bar{\Omega}_{mf} = \Omega_{mf} l/c_2$, $\bar{\Omega}_{tf} = \Omega_{tf} l/c_2$, $c_1 = \sqrt{(EA/\rho)}$, $c_2 = \sqrt{(T_0/\rho)}$, $\beta = c/c_2$, $\beta_1 = c_1/c_2$, $\bar{\eta}_u = C_u l/\rho c_2$, $\bar{\eta}_v = C_v l/\rho c_2$, $\bar{\eta}_w = C_w l/\rho c_2$, $\bar{\eta}_m = \sigma B_0^2 l/\rho c_2$, one can write the dimensionless governing equations for the moving string system as

$$\begin{aligned} &U_{\tau\tau} + [\bar{\eta}_u + \bar{\eta}_m(1 + \bar{B}\cos \bar{\Omega}_{mf}\tau)^2](U_\tau + \beta U_\xi) + 2\beta U_{\xi\tau} + [\beta^2 - \beta_1^2 - (1 + \bar{T}\cos \bar{\Omega}_{tf}\tau)]U_{\xi\xi} \\ &\quad - \beta_1^2(3U_\xi U_{\xi\xi} + V_\xi V_{\xi\xi} + W_\xi W_{\xi\xi} + \tfrac{3}{2}U_{\xi\xi}U_\xi^2 + \tfrac{1}{2}U_{\xi\xi}V_\xi^2 + \tfrac{1}{2}U_{\xi\xi}W_\xi^2 + U_\xi V_\xi V_{\xi\xi} \\ &\quad + U_\xi W_\xi W_{\xi\xi}) = -\bar{\eta}_m\beta(1 + \bar{B}\cos \bar{\Omega}_{mf}\tau)^2, \end{aligned} \tag{2a}$$

$$\begin{aligned} &V_{\tau\tau} + \bar{\eta}_v(V_\tau + \beta V_\xi) + 2\beta V_{\xi\tau} + [\beta^2 - 1 - \bar{T}\cos \bar{\Omega}_{tf}\tau]V_{\xi\xi} \\ &\quad - \beta_1^2(V_\xi U_{\xi\xi} + U_\xi V_{\xi\xi} + \tfrac{1}{2}V_{\xi\xi}U_\xi^2 + \tfrac{3}{2}V_{\xi\xi}V_\xi^2 + \tfrac{1}{2}V_{\xi\xi}W_\xi^2 + V_\xi U_\xi U_{\xi\xi} + V_\xi W_\xi W_{\xi\xi}) = 0, \end{aligned} \tag{2b}$$

$$W_{\tau\tau} + [\bar{\eta}_w + \bar{\eta}_m(1 + \bar{B}\cos \bar{\Omega}_{mf}\tau)^2](W_\tau + \beta W_\xi) + 2\beta W_{\xi\tau} + [\beta^2 - 1 - \bar{T}\cos \bar{\Omega}_{tf}\tau]W_{\xi\xi}$$

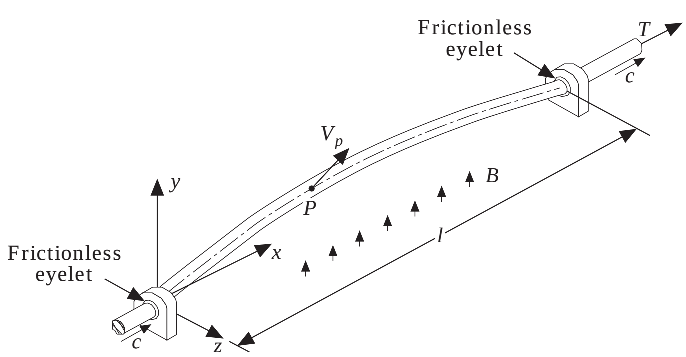

Figure 1. The string system subjected to magnetic field in the y direction and time-dependent tension in the x direction.

$$-\beta_1^2(W_\xi U_{\xi\xi} + U_\xi W_{\xi\xi} + \tfrac{1}{2}W_{\xi\xi}U_\xi^2 + \tfrac{1}{2}W_{\xi\xi}V_\xi^2 + \tfrac{3}{2}W_{\xi\xi}W_\xi^2 + W_\xi U_\xi U_{\xi\xi} + V_\xi W_\xi V_{\xi\xi}) = 0, \tag{2c}$$

and the dimensionless boundary conditions

$$U(0,\tau) = V(0,\tau) = W(0,\tau) = 0, \qquad U(1,\tau) = V(1,\tau) = W(1,\tau) = 0. \tag{3a, b}$$

From the above governing equations (2a–c), the following observations are made: (i) In the absence of the magnetic field, the system is the same as that in Huang *et al*. [1]. (ii) The Coriolis terms, $2\beta U_{\xi\tau}$, $2\beta V_{\xi\tau}$ and $2\beta W_{\xi\tau}$, appear in three equations (2a–c) for the moving string. (iii) It is seen that the alternating uniform transverse magnetic force, $\bar{\eta}_m(1+\bar{B}\cos\bar{\Omega}_{mf}\tau)^2$ appears in the equations (2a) and (2c) (x and z directions), but physically it acts in the y direction. Thus, the vibration in the y direction will not be affected by the magnetic field. (iv) The magnetic force acts as a damping effect in the x and z directions. (v) The nonhomogeneous term, $-\bar{\eta}_m\beta(1+\bar{B}\cos\bar{\Omega}_{mf}\tau)^2$, appears in the x direction, and this term is proportional to the moving speed parameter β.

3. PERTURBATION ANALYSIS

In order to solve these governing equations (2a–c), Galerkin's method is used here to discretize these governing equations. One can obtain three sets of ordinary differential equations with respect to three axes:

$$\ddot{e}_m + \sum_{n=1}^{\infty}[a_{mn}\dot{e}_n + b_{mn}e_n] - \beta_1^2\left\{\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}[D_{mni}(3e_ne_i + f_nf_i + g_ng_i)]\right.$$

$$\left. + \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[E_{mnij}(\tfrac{3}{2}e_ne_ie_j + \tfrac{1}{2}f_nf_ie_j + \tfrac{1}{2}g_ng_ie_j + e_nf_if_j + e_ng_ig_j)]\right\} = d_0, \tag{4a}$$

$$\ddot{f}_m + \sum_{n=1}^{\infty}[d_{mn}\dot{f}_n + h_{mn}f_n] - \beta_1^2\left\{\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}[D_{mni}(f_ne_i + e_nf_i)]\right.$$

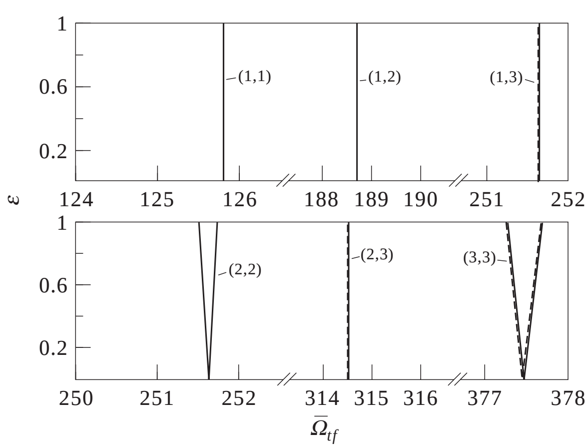

Figure 2. The effect of mode number on the stable-unstable regions in the x direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. ——, Three modes; ----, 13 modes.

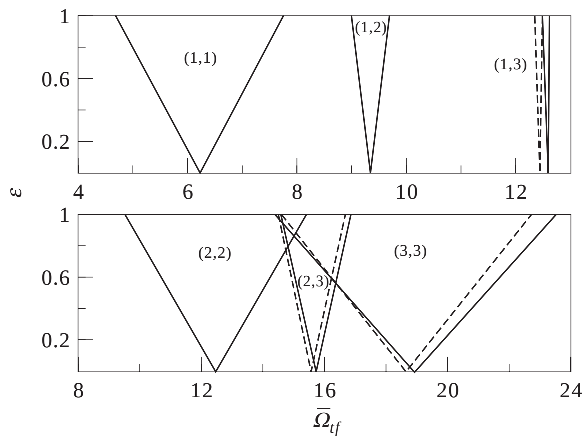

Figure 3. The effect of mode number on the stable-unstable regions in the y direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. ——, Three modes; ----, 13 modes.

$$+\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[E_{mnij}\left(\tfrac{1}{2}e_ne_if_j+\tfrac{3}{2}f_nf_if_j+\tfrac{1}{2}g_ng_if_j+f_ne_ie_j+f_ng_ig_j\right)\right]\Bigg\}=0, \quad (4b)$$

$$\ddot{g}_m+\sum_{n=1}^{\infty}[k_{mn}\dot{g}_n+l_{mn}g_n]-\beta_1^2\Bigg\{\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}[D_{mni}(g_ne_i+e_ng_i)]$$

$$+\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[E_{mnij}\left(\tfrac{1}{2}e_ne_ig_j+\tfrac{1}{2}f_nf_ig_j+\tfrac{3}{2}g_ng_ig_j+g_ne_ie_j+g_nf_if_j\right)\right]\Bigg\}=0, \quad (4c)$$

where

$$a_{mn}=(\bar{\eta}_u+\bar{\eta}_m(1+\overline{B}\cos\overline{\Omega}_{mf}\tau)^2)A_{mn}+2\beta B_{mn},$$
$$b_{mn}=(\beta\bar{\eta}_u+\beta\bar{\eta}_m(1+\overline{B}\cos\overline{\Omega}_{mf}\tau)^2)B_{mn}+[\beta^2-\beta_1^2-(1+\overline{T}\cos\overline{\Omega}_{tf}\tau)C_{mn},$$

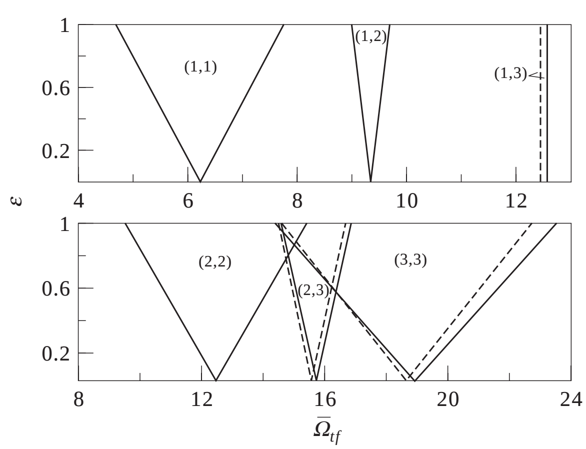

Figure 4. The effect of mode number on the stable-unstable regions in the z direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. ——, Three modes; ----, 13 modes.

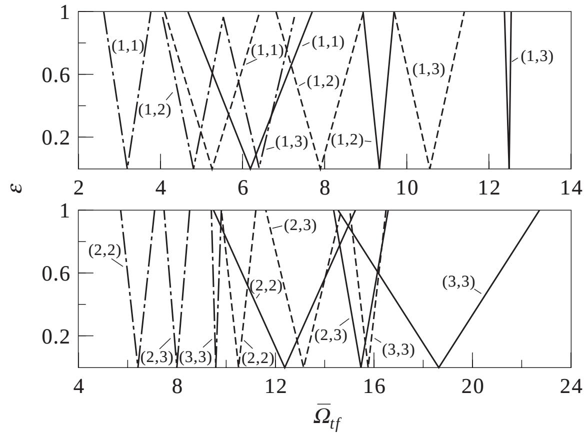

Figure 5. The effect of the transport speed parameter β on the stable-unstable regions in the y direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. ——, $\beta = 0{\cdot}1$; ----, $\beta = 0{\cdot}4$; –·–·, $\beta = 0{\cdot}7$.

$$d_0 = -\int_0^1 \bar{\eta}_m(1 + \bar{B}\cos\bar{\Omega}_{mf}\tau)^2\phi_m\,\mathrm{d}\xi, \qquad d_{mn} = \bar{\eta}_v A_{mn} + 2\beta B_{mn},$$

$$h_{mn} = \beta\bar{\eta}_v B_{mn} + (\beta^2 - 1 - \bar{T}\cos\bar{\Omega}_{tf}\tau)C_{mn},$$

$$k_{mn} = (\bar{\eta}_w + \bar{\eta}_m(1 + \bar{B}\cos\bar{\Omega}_{mf}\tau)^2)A_{mn} + 2\beta B_{mn},$$

$$l_{mn} = (\beta\bar{\eta}_w + \beta\bar{\eta}_m(1 + \bar{B}\cos\bar{\Omega}_{mf}\tau)^2)B_{mn} + (\beta^2 - 1 - \bar{T}\cos\bar{\Omega}_{tf}\tau)C_{mn},$$

$$A_{mn} = \int_0^1 \phi_n\phi_m\,\mathrm{d}\xi = \begin{cases} 1, & m = n \\ 0, & m \neq n, \end{cases}$$

$$B_{mn} = \int_0^1 \phi_n'\phi_m\,\mathrm{d}\xi = \begin{cases} 0, & \\ \dfrac{2nm(\cos m\pi\cos n\pi - 1)}{n^2 - m^2}, & m \neq n, \end{cases}$$

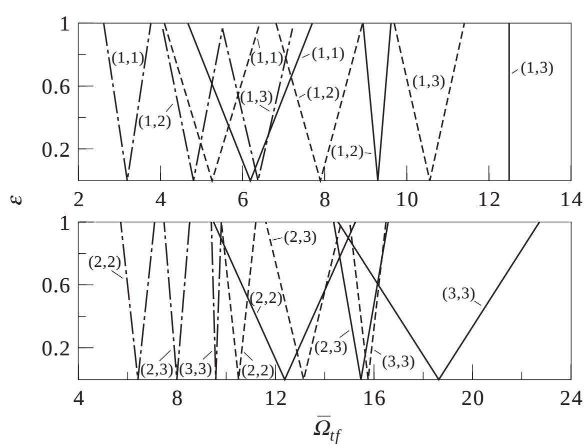

Figure 6. The effect of the transport speed parameter β on the stable-unstable regions in the z direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. ——, $\beta = 0{\cdot}1$; ----, $\beta = 0{\cdot}4$; –·–·, $\beta = 0{\cdot}7$.

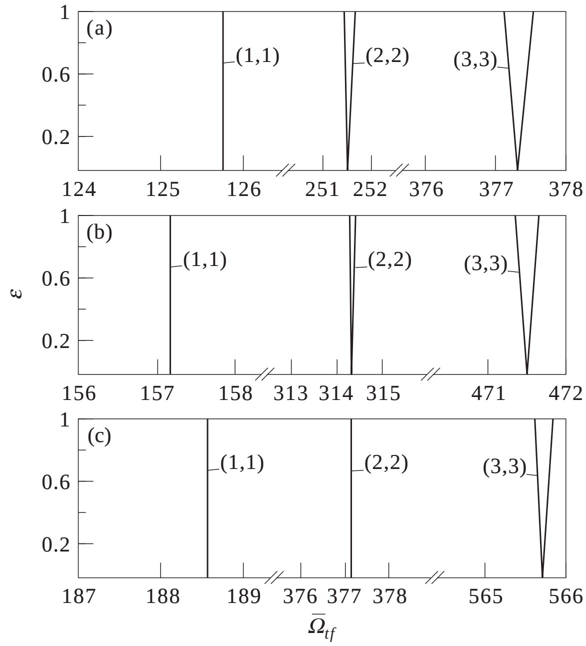

Figure 7. The effect of the wave propagation speed β_1 on the stable-unstable regions in the x direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. (a) $\beta_1 = 20$. (b) $\beta_1 = 25$. (c) $\beta_1 = 30$.

$$C_{mn} = \int_0^1 \phi_n'' \phi_m \, \mathrm{d}\xi = \begin{cases} -(n\pi)^2, & m = n \\ 0, & m \neq n, \end{cases}$$

$$D_{mni} = \int_0^1 \phi_n' \phi_i'' \phi_m \, \mathrm{d}\xi = \begin{cases} -\left(\dfrac{\sqrt{2}}{2}\right) ni^2\pi^3, & m = \pm n + i \\ & m = \pm n - i \\ \left(\dfrac{\sqrt{2}}{2}\right) ni^2\pi^3, & \\ 0, & \text{otherwise,} \end{cases}$$

$$E_{mnij} = \int_0^1 \phi_n' \phi_i' \phi_j'' \phi_m \, \mathrm{d}\xi = \begin{cases} -\frac{1}{2} nij^2\pi^4, & m = \pm n \pm i + j \\ \frac{1}{2} nij^2\pi^4, & m = \pm n \pm i - j \\ 0, & \text{otherwise.} \end{cases}$$

One defines both the tension ratio $\bar{T}$ and the magnetic ratio $\bar{B}$ to be the first order of the small parameter ε, that is $\bar{T} = O(\varepsilon)$ and $\bar{B} = O(\varepsilon)$, in the perturbation technique. The damping terms $\bar{\eta}_u = \varepsilon\eta_u$, $\bar{\eta}_v = \varepsilon\eta_v$ and $\bar{\eta}_w = \varepsilon\eta_w$ are assumed to have the same order as the first non-autonomous term. The magnetic constant $\bar{\eta}_m = \varepsilon\eta_{mf}$ is also assumed to be a first order term. Following the earlier work [1], a special modal analysis is used to decouple the discrete gyroscopic system and then the multiple-scale method is employed to approximate. The system responses and the expressions for the boundaries of the unstable regions are obtained.

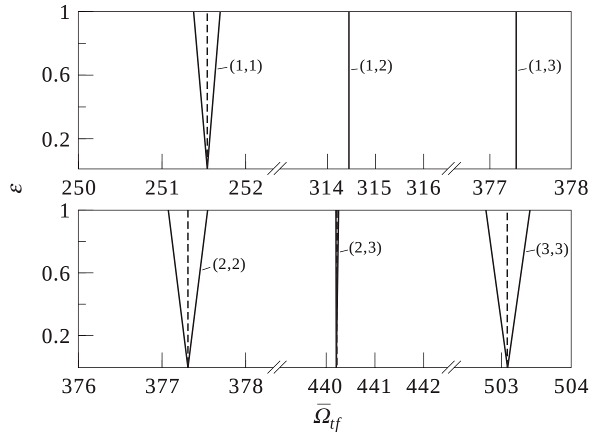

Figure 8. The effect of magnetic field on the stable-unstable regions in the x direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. (——), without magnetic field $\eta_{mf} = 0$; (- - -), without magnetic field $\eta_{mf} + 0.5$.

4. NUMERICAL RESULTS

From the equations of motion, the natural frequencies of the transverse vibration are $\Omega_m = m\pi\sqrt{1-\beta^2}$, $0 \leqslant \beta \leqslant 1$, $m = 1, 2, \ldots$. In this paper, the effects of various parameters, including the parameter β of the string transport speed, the parameter β_1 of the wave propagation speed, and the external magnetic field η_{mf} on the dynamic stability will be analyzed.

Figures 2–4 show the effect of the mode numbers on the dynamic stability in the x, y and z directions. The other parameters are $\beta = 0{\cdot}1$, $\beta_1 = 20$, $\eta_{mf} = 0{\cdot}1$, and $\eta_u = \eta_v = \eta_w = 0{\cdot}01$. In these figures, ($p$, q) denotes the resonance near $\Omega_p + \Omega_q$. It can be seen in these figures that the unstable regions are shifted to the lower frequencies when the mode number 13 is used. In the present simulations, when the mode number is larger than 13, which is not shown here, the unstable regions are almost the same with those of the mode number 13. Hence, in the following cases presented in this paper, the mode number 13 is used.

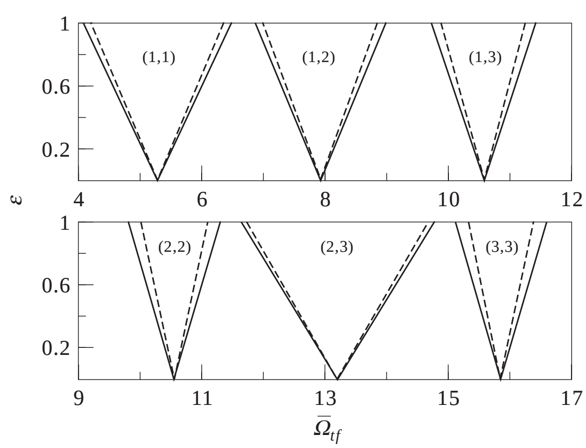

Figure 9. The effect of magnetic field on the stable-unstable regions in the z direction where (p, q) denotes the resonance near $\Omega_p + \Omega_q$. (——), without magnetic field $\eta_{mf} = 0$; (- - -), with magnetic field $\eta_{mf} = 0.5$.

Figures 5–6 present the stable–unstable regions in the y and z directions for the different values of the transport speed parameter $\beta = 0{\cdot}1$, $0{\cdot}4$ and $0{\cdot}7$. The other parameters are $\beta_1 = 20$, $\eta_{mf} = 0{\cdot}1$, and $\eta_u = \eta_v = \eta_w = 0{\cdot}01$. In these figures, the stable-unstable regions are developed for the summed-type resonances. It is found that the unstable regions will be shifted to the lower frequencies when the string transport speed parameter β increases. Therefore, the string has the unstable region in the lower frequency domain when its transport speed is higher.

The factors including damping coefficient, natural frequency and magnetic field will affect the unstable regions of the system. The location of the unstable region will be near the sum or the difference of any two natural frequencies of the string system. Moreover, the natural frequencies vary in accordance with the change in the string transport speed parameter β and the wave propagation speed parameter β_1. Therefore, as these parameters vary, the location of the unstable region will be shifted simultaneously.

The metal strings with the wave speed ratio $c_1^2/c_2^2 = EA/T = \mathrm{O}$ (400–1000) are considered [8]. The rigidity of the string will dominate the natural frequency of the system. The higher rigidity of the string has a higher natural frequency and a smaller unstable region. The stable-unstable regions in the y and z directions will not be affected by the parameters β_1. In Figures 7(a–c), the stable–unstable regions with various values of the wave propagation speed parameter β_1 in the x direction are shown. The other parameters are $\beta = 0{\cdot}4$, $\eta_{mf} = 0{\cdot}1$ and $\eta_u = \eta_v = \eta_w = 0{\cdot}01$. From these figures, it is found that as the wave propagation speed parameter β_1 increases, the unstable regions will be shifted toward higher frequencies and the system becomes more stable.

From equation (2b), the unstable regions in the y direction are not affected by the external magnetic field in the same direction. Figures 8 and 9 show that as the magnetic field increases, the unstable regions in the x and z directions will become small or even disappear. In these figures, the other parameters are $\beta = 0{\cdot}4$, $\beta_1 = 20$ and $\eta_u = \eta_v = \eta_w = 0{\cdot}01$.

5. CONCLUSIONS

The dynamic stability of an axially moving string with both actions of an alternating uniform transverse magnetic field and a periodic tension force is investigated in this paper. From previous studies, some conclusions are drawn.

(1) It is seen that the unstable regions in the y direction will not be affected by the external magnetic field in the same direction.

(2) The transverse magnetic field causes the unstable regions to be smaller.

(3) The unstable regions will be shifted to the lower frequencies as the string transport speed parameter increases or the wave propagation speed parameter decreases.

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